Dimer models and homological mirror symmetry for triangles

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Abstract

We prove a conjecture on the relation between dimer models, coamoebas and vanishing cycles for the mirrors of two-dimensional toric Fano stacks of Picard number one. As a corollary, we obtain a torus-equivariant version of homological mirror symmetry for such stacks.

1 Introduction

With a convex lattice polygon $\Delta \subset \mathbb{R}^2$ containing the origin in its interior, one can associate a directed A_∞-category in three different ways:

- Let
$$W(x,y) = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j$$
be a Laurent polynomial whose Newton polygon coincides with Δ;
$$\Delta = \operatorname{Conv}\{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\}.$$
If the coefficients a_{ij} are sufficiently general, then W defines an exact symplectic Lefschetz fibration with respect to the cylindrical Kähler form on $(\mathbb{C}^\times)^2$, and one can associate the directed Fukaya category $\mathfrak{Fuk}\, W$ whose set of objects is a distinguished basis of vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes [Sei01, Sei08].

- Let X be the two-dimensional toric Fano stack associated with the stacky fan whose one-dimensional cones are generated by vertices of Δ. The derived category $D^b \operatorname{coh} X$ of coherent sheaves on X has a full strong exceptional collection $(E_i)_i$ consisting of line bundles [BH09], which induces a derived equivalence with the category of finitely-generated modules over the total morphism algebra $\mathbb{C}\Gamma = \operatorname{End}(\bigoplus_i E_i)$. The full subcategory of the enhanced derived category of $\mathbb{C}\Gamma$ consisting of simple modules will be denoted by $\mathcal{C}$.

- Let (G, D) be a pair of a consistent dimer model and a perfect matching on it, whose characteristic polygon coincides with Δ. One can associate a directed A_∞-category $\mathcal{A}$ with such a pair, and there is a quasi-equivalence
$$\mathcal{A} \cong \mathcal{C}$$
of A_∞-categories for a suitable choice of an exceptional collection on X [IU, FU].

A dimer model is a bicolored graph on a real 2-torus which encodes the information of a quiver with potential. See e.g. [FU] and references therein for basic definitions on dimer models. In this paper, we deal only with hexagonal dimer models appearing in [UYb].

The following conjecture is motivated by [FHKV08]:

Conjecture 1.1 ([UYa, Conjecture 6.2])**.** *Let Δ be a lattice polygon containing the origin in its interior. Then for a suitable choice of*

- *a Laurent polynomial W whose Newton polygon coincides with Δ, and*
- *a distinguished basis of vanishing cycles on $W^{-1}(0)$,*

there is a bicolored graph Y on $W^{-1}(0)$ such that

- *an edge of Y corresponds to an intersection of vanishing cycles,*
- *a node of Y corresponds to a holomorphic disk bounded by vanishing cycles,*
- *the color of a node corresponds to the sign of the A_∞-operation determined by the disk,*
- *the image of Y by the argument map*

$$\begin{array}{cccc} \mathrm{Arg}: & (\mathbb{C}^\times)^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ & \cup & & \cup \\ & (x, y) & \mapsto & \dfrac{1}{2\pi}(\arg x, \arg y) \end{array}$$

  is a consistent dimer model G,
- *the order on the distinguished basis of vanishing cycle gives a perfect matching D, and*
- *the characteristic polygon of the pair (G, D) coincides with Δ.*

We prove the following in this paper:

Theorem 1.2. *Conjecture 1.1 holds if Δ is a triangle.*

As a corollary, one obtains a torus-equivariant version

$$D^b \operatorname{coh}^{\mathbb{T}} X \cong D^b \mathfrak{Fuk}\, \widetilde{W} \tag{1.1}$$

of homological mirror symmetry [Kon95, Kon98] for two-dimensional toric Fano stacks of Picard number one. Here $D^b \operatorname{coh}^{\mathbb{T}} X$ is the equivariant derived category of coherent sheaves on X with respect to the algebraic torus $\mathbb{T}$ acting on X, and $\widetilde{W}$ is the pull back of W to the universal cover of the torus. The non-equivariant version of (1.1) for weighted projective planes is due to [Sei01, AKO08].

The organization of this paper is as follows: In Section 2, we recall the construction of two-dimensional toric Fano stacks from lattice triangles and discuss its relation with weighted projective planes. In Section 3, we describe vanishing cycles of W following [AKO08] closely. In Section 4, we study the behavior of vanishing cycles under the argument map and prove Theorem 1.2.

2 Triangles and weighted projective planes

Let

$$\Delta = \operatorname{Conv}\{v_1, v_2, v_3\}$$

be a lattice triangle in $\mathbb{R}^2$ containing the origin in its interior. The toric Fano stack X associated with Δ is defined as the quotient stack

$$X = [(\mathbb{C}^3 \setminus 0)/K]$$

of the complement of the origin in $\mathbb{C}^3$ by the natural action of the kernel

$$K = \operatorname{Ker}(\phi \otimes \mathbb{C}^\times : (\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^2),$$

where

$$\begin{array}{cccc} \phi : & \mathbb{Z}^3 & \to & \mathbb{Z}^2 \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & e_i & \mapsto & v_i \end{array}$$

is the homomorphism of abelian groups sending the i-th coordinate vector e_i to the vertex v_i of Δ for $i = 1, 2, 3$.

If ϕ is surjective, then K is isomorphic to $\mathbb{C}^\times$ and the action of $\alpha \in \mathbb{C}^\times \cong K$ on $\mathbb{C}^3$ is given by $(x, y, z) \mapsto (\alpha^a x, \alpha^b y, \alpha^c z)$ for some relatively prime positive integers a, b, and c. The resulting stack $[(\mathbb{C}^3 \setminus 0)/K]$ is the weighted projective plane

$$X = \mathbb{P}(a, b, c),$$

and any weighted projective plane with relatively prime weights can be obtained in this way by setting ϕ to be the natural projection

$$\phi : \mathbb{Z}^3 \to \operatorname{coker}(\varphi) \cong \mathbb{Z}^2$$

to the cokernel of

$$\begin{array}{cccc} \varphi : & \mathbb{Z} & \to & \mathbb{Z}^3 \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & 1 & \mapsto & (a, b, c). \end{array}$$

If $d = \gcd(a, b, c) \neq 1$, then the derived category of coherent sheaves on $\mathbb{P}(a, b, c)$ is a direct sum

$$D^b \operatorname{coh} \mathbb{P}(a, b, c) \cong (D^b \operatorname{coh} \mathbb{P}(a', b', c'))^{\oplus d}, \qquad (a, b, c) = (a'd, b'd, c'd),$$

and the mirror of $\mathbb{P}(a, b, c)$ is a disjoint union of d copies of the mirror for $\mathbb{P}(a', b', c')$.

If the map ϕ is not surjective, then one can factor ϕ as

$$\phi = \phi_2 \circ \phi_1 : \mathbb{Z}^3 \xrightarrow{\phi_1} \mathbb{Z}^2 \xrightarrow{\phi_2} \mathbb{Z}^2$$

where ϕ_1 is the surjection to $\operatorname{Im}(\phi) \cong \mathbb{Z}^2$ and ϕ_2 is the inclusion of $\operatorname{Im}(\phi)$ to $\mathbb{Z}^2$. One obtains an exact sequence

$$1 \to K_1 \to K \to K_2 \to 1,$$

where $K_1 = \operatorname{Ker}(\phi_1 \otimes \mathbb{C}^\times)$ and $K_2 = \operatorname{Ker}(\phi_2 \otimes \mathbb{C}^\times)$, and $X = [(\mathbb{C}^3 \setminus 0)/K]$ is the quotient stack

$$X = [\mathbb{P}(a, b, c)/K_2]$$

for the weight (a, b, c) such that $\mathbb{P}(a, b, c) = [(\mathbb{C}^3 \setminus 0)/K_1]$.

3 Vanishing cycles for triangles

Let $\Delta \subset \mathbb{R}^2$ be a lattice triangle which contains the origin in its interior. One can choose an $SL_2(\mathbb{Z})$-transformation to set

$$\Delta = \mathrm{Conv}\{(a,0),(b,c),(-d,-e)\}$$

where a,c,d,e are positive and b is non-negative. Let

$$W = x^a + x^b y^c + \frac{1}{x^d y^e}$$

be a Laurent polynomial whose Newton polygon coincides with Δ and consider the diagram

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\varpi"'] \arrow[rr, bend left, "W = \psi \circ \varpi"] & \mathbb{C} \times \mathbb{C}^\times \arrow[r, "\psi"'] & \mathbb{C} \end{tikzcd}$$

where

$$\varpi(x,y) = (W(x,y), x)$$

and

$$\psi(w,x) = w.$$

For general $t \in \mathbb{C}$, the map

$$\mathcal{E}_t \xrightarrow{\varpi_t} \mathcal{S}_t$$

from $\mathcal{E}_t = W^{-1}(t)$ to $\mathcal{S}_t = \psi^{-1}(t)$ is a $(c+e)$-fold cover of $\mathcal{S}_t$, which can naturally be identified with the x-plane. The fiber of ϖ_t is defined by

$$x^{b+d} y^{c+e} + (x^a - t) x^d y^e + 1 = 0,$$

which can be written as

$$A y^{c+e} + B y^e + 1 = 0$$

where $A = x^{b+d}$ and $B = (x^a - t)x^d$. The critical points of ϖ_t are defined by

$$\begin{cases} A y^{c+e} + B y^e + 1 = 0, \\ (c+e) A y^{c+e-1} + e B y^{e-1} = 0. \end{cases} \tag{3.1}$$

By eliminating y from (3.1), one obtains

$$(-1)^g \frac{c^c e^e}{g^g} (x^a - t)^g x^h = 1$$

as the defining equation for the critical values of ϖ_t where $g = c+e$ and $h = cd - be$. The set

$$D(t) = \left\{ x \in \mathbb{C}^\times \;\middle|\; (-1)^g \frac{c^c e^e}{g^g} (x^a - t)^g x^h = 1 \right\}$$

consists of $ag + h$ points for general t, which becomes singular when

$$x^a = \frac{h}{ag+h} t$$

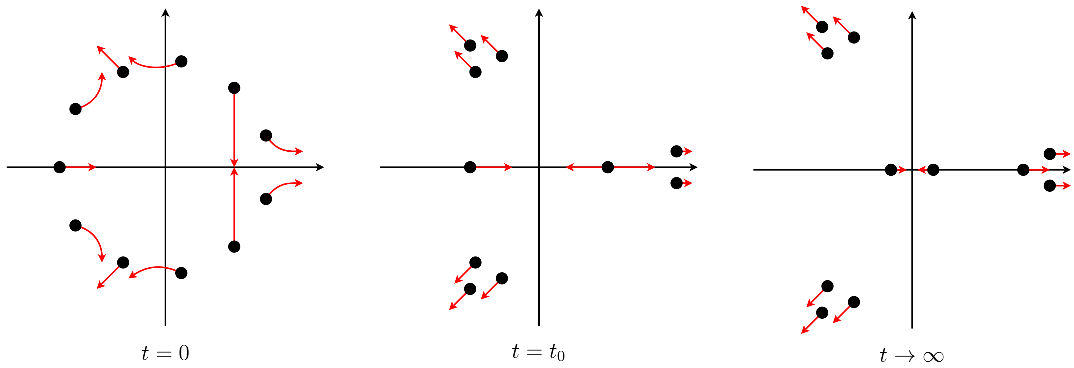

Figure 3.1: Branch points of ϖ_t for $(a, g, h) = (3, 3, 2)$

and

$$t^{g+\frac{h}{a}} = \frac{1}{c^c e^e}\left(1+\frac{h}{a}\right)^g \left(1+\frac{ag}{h}\right)^{\frac{h}{a}}. \tag{3.2}$$

Assume $k := \gcd(a, h) = 1$, so that (3.2) has $ag + h$ solutions, which is equal to the area $|\Delta|$ of Δ. The $k \neq 1$ case can be reduced to this case by the k-fold cover $x \mapsto x^k$ of the x-plane. The set of solutions of (3.2) is the set of critical values of W. We choose the straight line segments from the origin to the critical values of W as a distinguished set of vanishing paths. The corresponding vanishing cycles can be computed by studying the behavior of the branch points of ϖ_t along the vanishing paths. Let t_0 be the unique positive real critical value of W and consider the behavior of the branch points of ϖ_t as one varies t from zero to infinity along the positive real axis. Branch points at $t = 0$ are distributed on a circle centered at the origin, and their arguments are given by $\frac{(g+2n)\pi}{ag+h}$ for $n = 1, \dots, ag+h$. As t goes from zero to t_0, the branch points with arguments $\pm\frac{g\pi}{ag+h}$ come close to each other and merge on the real line. As t goes from t_0 to infinity, the merged branch points split into two again, and the whole set of branch points are divided into $a+1$ groups; one group consists of h branch points coming close to the origin, and each of the remaining a groups consists of g branch points going off to infinity. Figure 3.1 shows this behavior for $(a, g, h) = (3, 3, 2)$. It follows that the vanishing cycle $C_0 \subset W^{-1}(0)$ of W along the straight line segment from the origin to t_0 lies above the matching path obtained as the trajectory of two branch points of ϖ_t whose arguments are $\pm\frac{g\pi}{ag+h}$ at $t = 0$.

Let $\phi_0 : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the linear map represented by the matrix

$$\begin{pmatrix} a+d & b+d \\ e & c+e \end{pmatrix},$$

and

$$K_0 = \operatorname{Ker}(\phi_0 \otimes \mathbb{C}^\times)$$

be the kernel of the homomorphism

$$\phi_0 \otimes \mathbb{C}^\times : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2.$$

Then an element $(\alpha, \beta) \in K_0$ gives a map

$$\begin{array}{ccc} W^{-1}(t) & \to & W^{-1}(\alpha^d \beta^e t) \\ \cup & & \cup \\ (x, y) & \mapsto & (\alpha x, \beta y), \end{array}$$

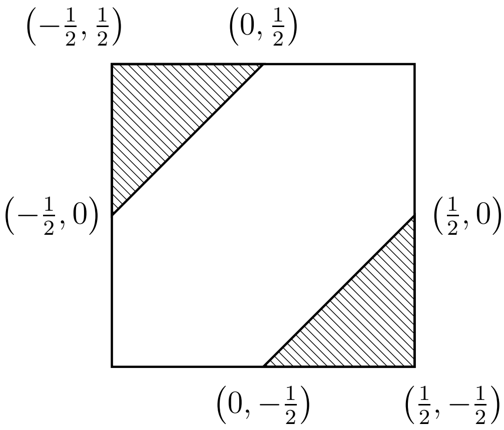

Figure 4.1: The coamoeba of $x + y + 1 = 0$

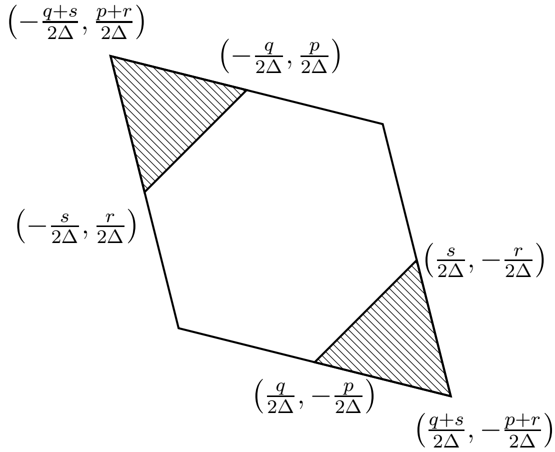

Figure 4.2: A part of the coamoeba of $W^{-1}(0)$

which induces a free transitive action of K_0 on the distinguished basis of vanishing cycles on $W^{-1}(0)$ along straight line segments from the origin to critical values of W.

4 Coamoebas and vanishing cycles

Recall from [UYb, Theorem 7.1] that the coamoeba of

$$W^{-1}(0) = \{(x, y) \in (\mathbb{C}^\times)^2 \mid 1 + x^{a+d}y^e + x^{b+d}y^{c+e} = 0\}$$

is given by the pull-back of the coamoeba of

$$\{(x, y) \in (\mathbb{C}^\times)^2 \mid 1 + x + y = 0\}$$

shown in Figure 4.1 by the map $\psi \otimes (\mathbb{R}/\mathbb{Z}) : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, where $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the linear map represented by the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+d & e \\ b+d & c+e \end{pmatrix}.$$

Figure 4.2 shows a part of the coamoeba, and the entire coamoeba is obtained by gluing $|\Delta| = ps - rq$ copies of it.

The coamoeba is the union of open triangles and their vertices, and the inverse image of the set of vertices of the coamoeba of $1 + x + y = 0$ is the real part of $1 + x + y = 0$. It is parametrized as

$$\begin{cases} x = t, \\ y = -t - 1, \end{cases}$$

and divided into three parts

$$t < -1, \quad -1 < t < 0, \quad \text{and} \quad t > 0.$$

It follows that the inverse images of vertices of the coamoeba of $W^{-1}(0)$ is parametrized as

$$(\psi \otimes \mathbb{C}^\times)^{-1}(t, -1 - t).$$

Since ψ^{-1} is given by the matrix

$$\frac{1}{|\Delta|}\begin{pmatrix} c+e & -e \\ -b-d & a+d \end{pmatrix},$$

the x-projection of the inverse images of vertices of the coamoeba is parametrized as

$$x^{|\Delta|} = \frac{t^{c+e}}{(-1-t)^e}.$$

By studying the behavior of the function

$$f(t) = \frac{t^{c+e}}{(-1-t)^e},$$

one can see that the x-projections of the inverse images of vertices of the coamoeba corresponding to the vertex $(\frac{1}{2}, 0)$ of the coamoeba of $x+y+1$ are half lines from the branch points of ϖ_0 to infinity with constant arguments. The x-projections of inverse images of other vertices of the coamoeba are half lines from the origin to infinity with constant arguments. The fiber $W^{-1}(0)$ is obtained by gluing $c+e$ copies of the x-plane which are cut into $2|\Delta|$ pieces along these half lines.

Now consider six triangles in Figure 4.5 which are adjacent to two triangles in Figure 4.2. The corresponding pieces of the copies of the x-plane are shown in Figure 4.3 and Figure 4.4. The discussion in Section 3 shows that the vanishing cycle $C_0 \subset W^{-1}(0)$ along the straight line segment from the origin to the positive real critical value is obtained by gluing the curves in Figure 4.3 and Figure 4.4 connecting branch points with arguments $\pm\frac{s}{\Delta}\pi$. The argument projection of C_0 is shown in Figure 4.5, which naturally corresponds to a face of the hexagonal dimer model G shown in Figure 4.6. Other vanishing cycles are obtained from C_0 by the action of K_0 as described in Section 3, so that the argument projection induces a natural bijection between the distinguished basis of vanishing cycles of W and the set of faces of G. One can easily see that an edge of G corresponds to an intersection of vanishing cycles under this bijection, and a node of G gives a holomorphic triangle which contributes to the A_∞-operation $\mathfrak{m}_2$ on the Fukaya category. When X is a weighted projective plane, a comparison with the discussion in [AKO08] shows that the color of the node matches the sign in the A_∞-operation, and the ordering on the distinguished basis of vanishing cycles defines an internal perfect matching of G. The toric Fano stack associated with a general lattice triangle can be obtained from the weighted projective plane as a toric orbifold, and Theorem 1.2 is proved.

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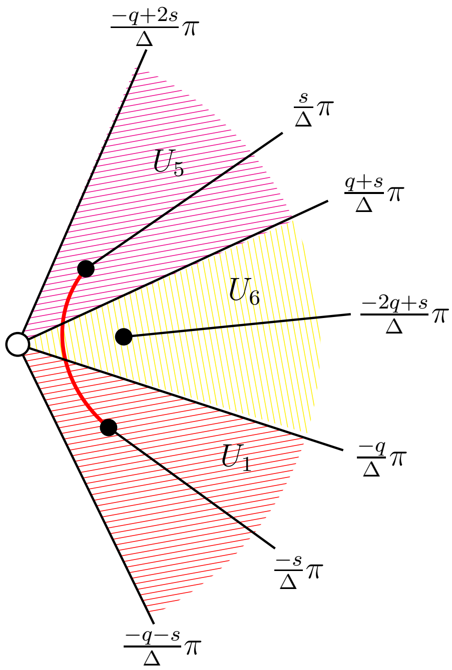

Figure 4.3: Three triangles and a part of the vanishing cycle C_0

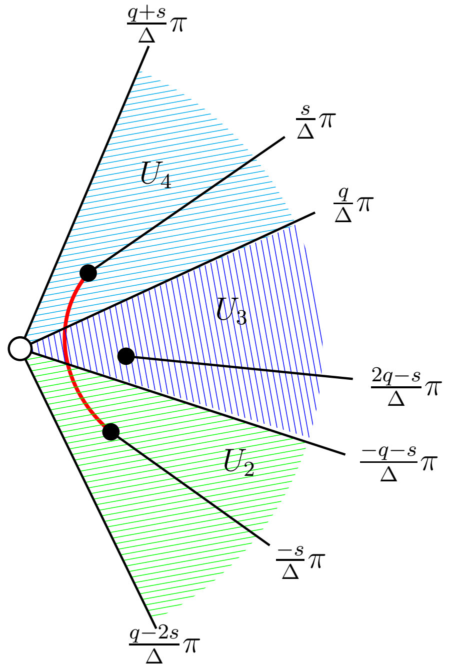

Figure 4.4: Three other triangles and the other part of the vanishing cycle C_0

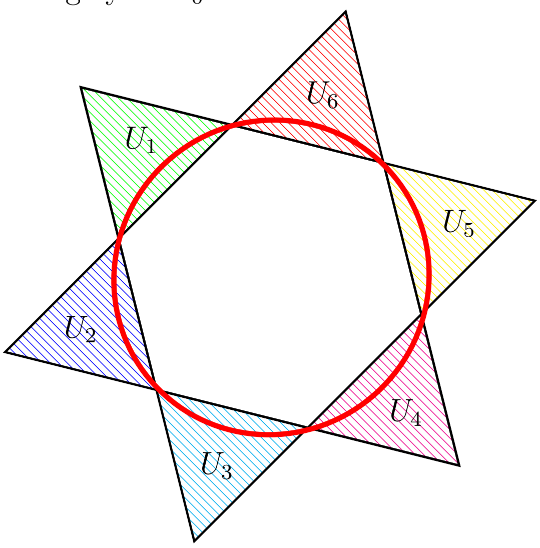

Figure 4.5: The vanishing cycle C_0 on the coamoeba

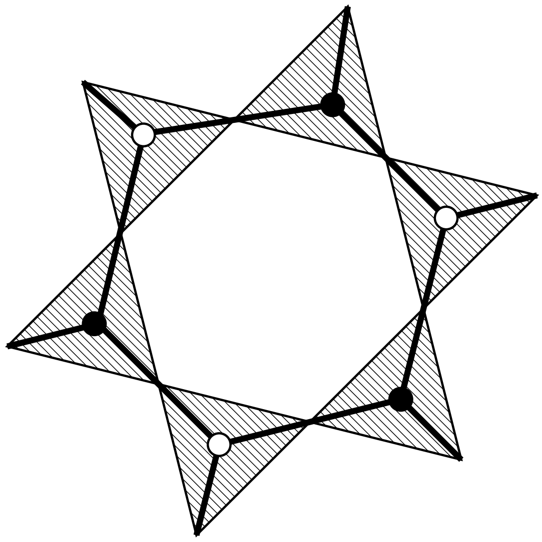

Figure 4.6: The dimer model

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